MATH 2050 C Lecture 18 (Mar 22)

Midterm graded and returned via email

Statistics: mean = 55.4, SD = 10.7, Highest = 70

Recall: $f: A \to \mathbb{R}$, $c \in \mathbb{R}$ cluster pt. of A

$$\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - L| < \varepsilon \text{ when } x \in A,\ 0 < |x - c| < \delta.$$

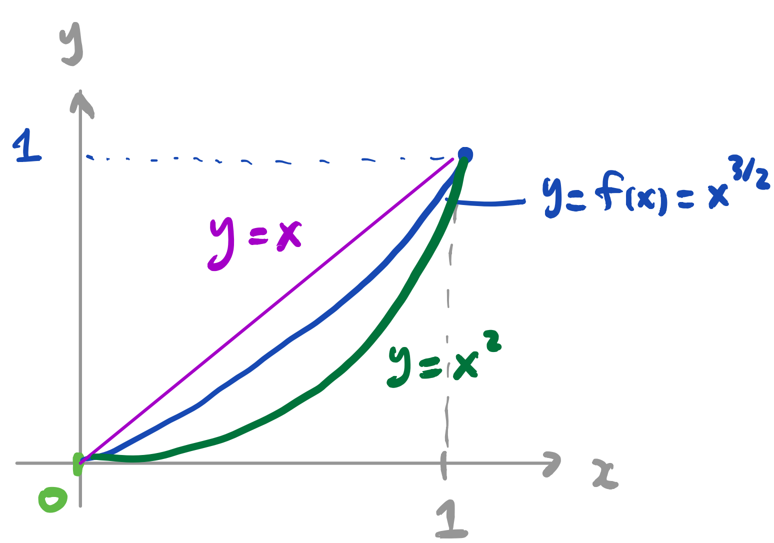
Sequential Criteria, Divergence criteria, Limit theorems ...

Sandwich / Squeeze Thm ...

Example 1: $\boxed{\lim_{x \to 0} x^{3/2} = 0}$

Proof: $f: A = \{x \in \mathbb{R} \mid x \geq 0\} \to \mathbb{R}$; $f(x) := x^{3/2}$

and 0 is a cluster pt.



Observe that

$$x^2 \leq x^{3/2} \leq x$$

for all $x \in [0, 1]$

Since $\lim_{x \to 0} x^2 = 0 = \lim_{x \to 0} x$,

by Squeeze thm,

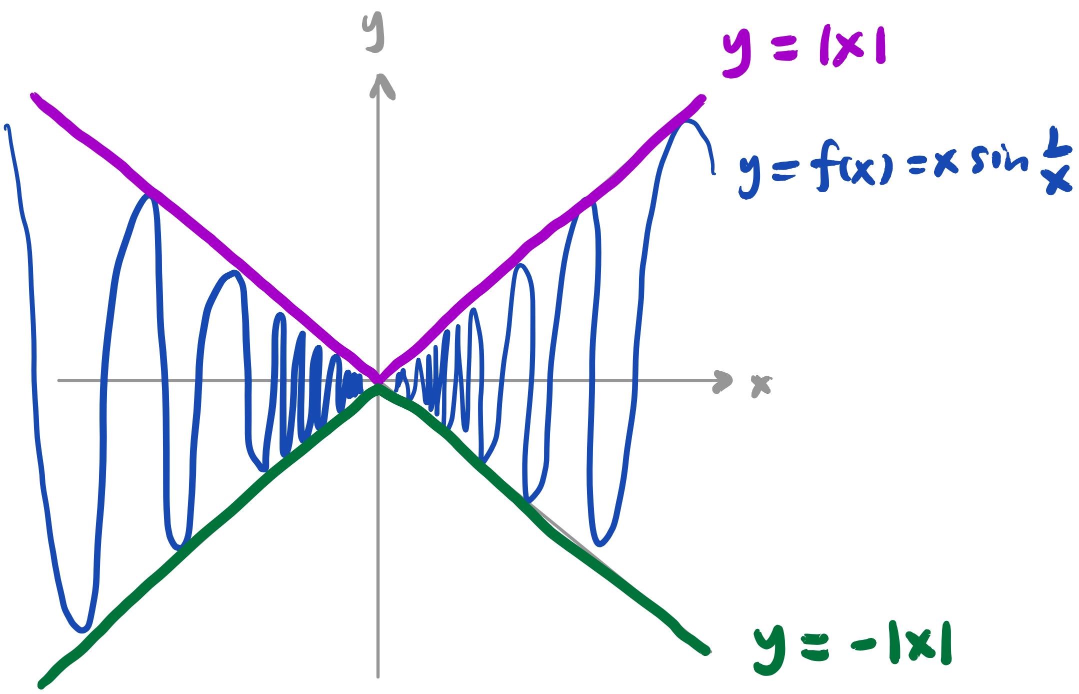
$$\lim_{x \to 0} x^{3/2} = 0$$

$\square$

Example 2: $\boxed{\lim_{x\to 0} x \sin\frac{1}{x} = 0}$

Recall: $\lim_{x\to 0} \sin\frac{1}{x}$ does NOT exist.

Proof: $f: A = \{x\in\mathbb{R} \mid x \neq 0\} \to \mathbb{R}$; $f(x) := x\sin\frac{1}{x}$.



Observe that since $\left|\sin\frac{1}{x}\right| \leq 1$

$$-|x| \leq x\sin\frac{1}{x} \leq |x| \qquad \forall\, x\in\mathbb{R},\ x\neq 0.$$

Notice that $\lim_{x\to 0} |x| = 0 = \lim_{x\to 0} -|x|$, by Sandwich theorem,

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0 .$$

$\blacksquare$

Recall: Suppose $\lim(x_n) = L > 0$. Then.

$\exists\, K \in \mathbb{N}$ s.t $x_n > 0 \quad \forall n \geq K$

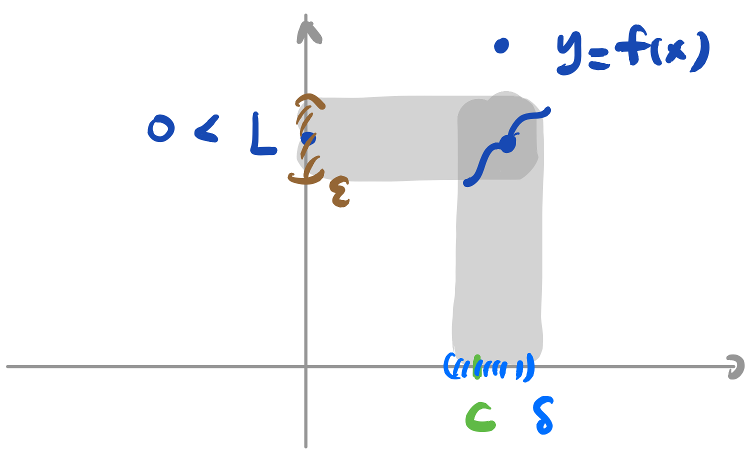
Prop: Suppose $\lim\limits_{x \to c} f(x) = L > 0$. Then $\exists\, \delta > 0$ s.t.

$f(x) > 0$ when $x \in A$, $0 < |x - c| < \delta$

Proof: Take $\varepsilon = \frac{L}{2} > 0$, by ε-δ def^{n}. $\exists\, \delta = \delta(\frac{L}{2}) > 0$

s.t $\quad |f(x) - L| < \frac{L}{2} \quad \forall x \in A,\ 0 < |x - c| < \delta$

$\Rightarrow \quad f(x) \geq L - \frac{L}{2} = \frac{L}{2} > 0 \quad \forall x \in A,\ 0 < |x - c| < \delta$



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Remark: The above Prop does NOT hold if $L \geq 0$.

Recap: $\mathbb{R}$ ⇝ $\lim(x_n)$ ⇝ $\lim_{x \to c} f(x)$ ⇝ "continuity"

Ch.2   Ch.3   Ch.4   Ch.5

Continuity of functions

Q: What does "continuity" mean?

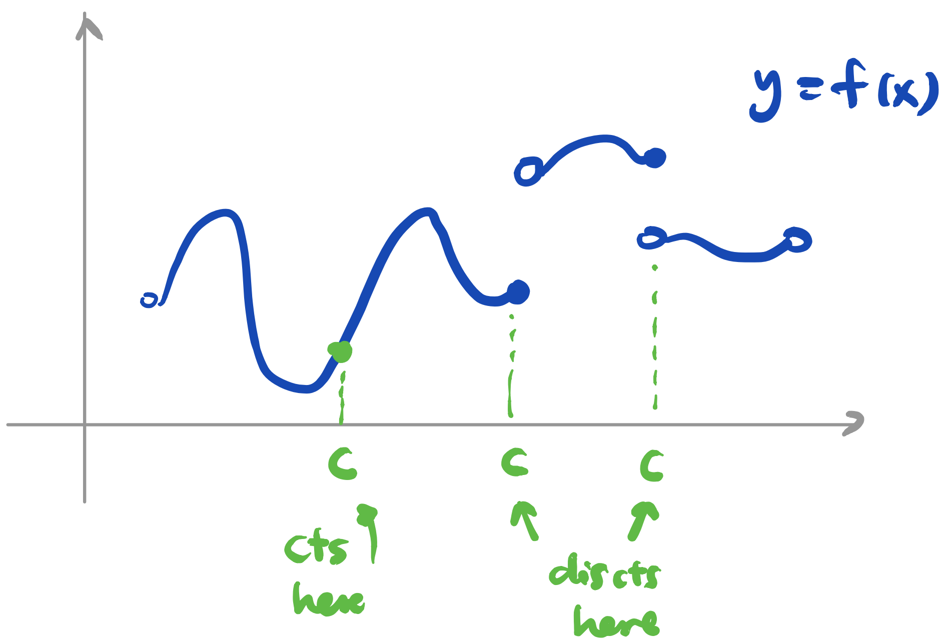
$f : A \to \mathbb{R}$

"A: "f is cts at c"

$\iff$ " $f(x) \overset{\varepsilon}{\approx} f(c)$ when $x \overset{\delta}{\approx} c$ "

Note: $c \in A$ so that $f(c)$ is defined!

Picture



Def$\underline{\text{n}}$: (ε-δ def$\underline{\text{n}}$ for continuity)

Given $f : A \to \mathbb{R}$ and $\boxed{c \in A}$, we say that

" f is continuous (cts) at c " if

$\forall\, \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever } x \in A \text{ and } |x - c| < \delta$$

Remark: Compare to the ε-δ def$\underline{\text{n}}$ of $\lim_{x \to c} f(x)$,

(1) The limit L is replaced by $f(c)$

So, $c \in A$. And $f(c)$ matters.

(2) We don't have to write $0 < |x - c| < \delta$

$|f(x) - f(c)| < \varepsilon$ always hold at $x = c$

(3) $c \in A$ may or may not be a cluster pt. here.

For the last point (3), let's us take a closer look.

interesting case

Case 1: $c \in A$ is a cluster pt. of A.

"f is cts at $c \in A$" $\iff$ "$\lim_{x \to c} f(x) = f(c)$"

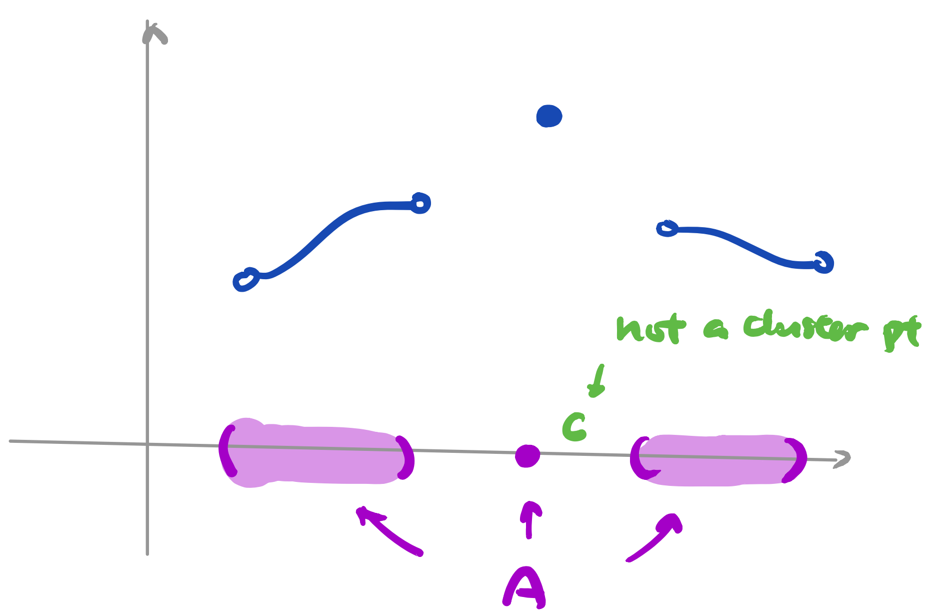
& so, "continuity at c" means you can evaluate the limit at c by "substitution"

Case 2: $c \in A$ is NOT a cluster pt of A.

THEN, ANY f is cts at $c \in A$.

Why? $\exists\, \delta > 0$ st $A \cap (c - \delta, c + \delta) = \{c\}$

then the ε-δ def? is always satisfied.



Def^{n}: $f: A \to \mathbb{R}$ is cts on a subset $B \subseteq A$ if f is cts at **EVERY** $c \in B$

In particular, if $B = A$, then we just say that $f: A \to \mathbb{R}$ is cts (everywhere).

Examples of continuous functions

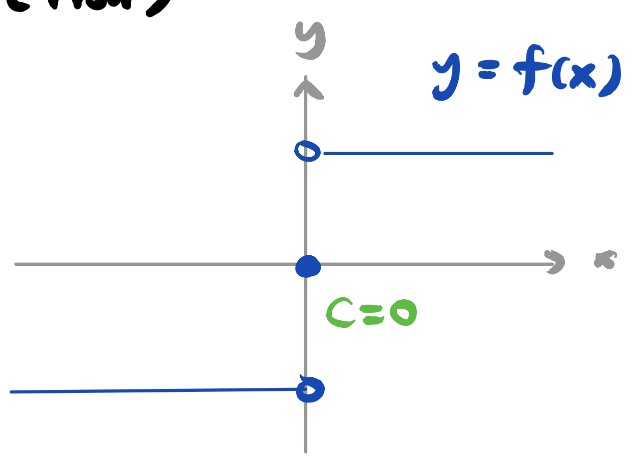
- $f(x) = b$ constant function
- $f(x) = x$ or x^2
- $f(x) =$ polynomial in x
- $f(x) = \sin x$ or $\cos x$ or $\tan x$
- $f(x) = e^x$ or $\sqrt{x}$

Examples of dis-continuous functions

Example 1: (The sign function)

$f: A = \mathbb{R} \to \mathbb{R}$

$$f(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$



Show that f is NOT cts at $c=0$.

Proof: Note that $c=0$ is a cluster pt. of $A=\mathbb{R}$.

Check if $\lim_{x\to 0} f(x) \overset{?}{=} f(0)$ $(=0)$

Claim: $\lim_{x\to 0} f(x)$ does NOT exist!

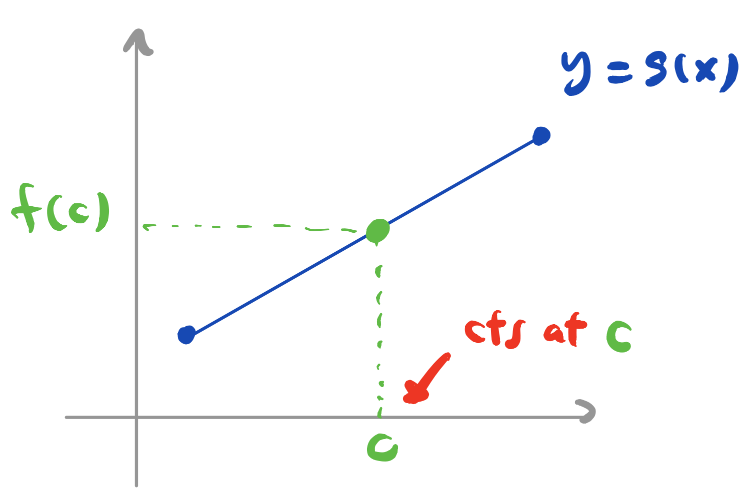
Consider a seq. $(x_n) = \left(\frac{(-1)^n}{n}\right) \to 0$

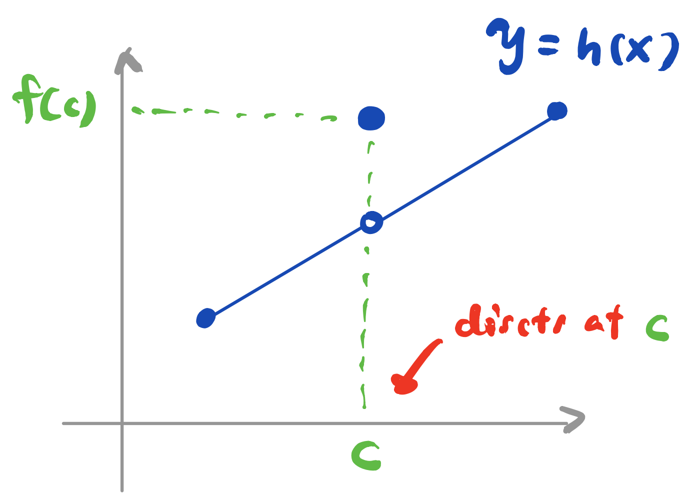
but $(f(x_n)) = ((-1)^n)$ is divergent

By Divergence Criterion, then $\lim_{x\to 0} f(x)$ does NOT exist

$\blacksquare$

Remark: It doesn't matter what is the value of $f(0)$ in this example. But sometimes it does matter.





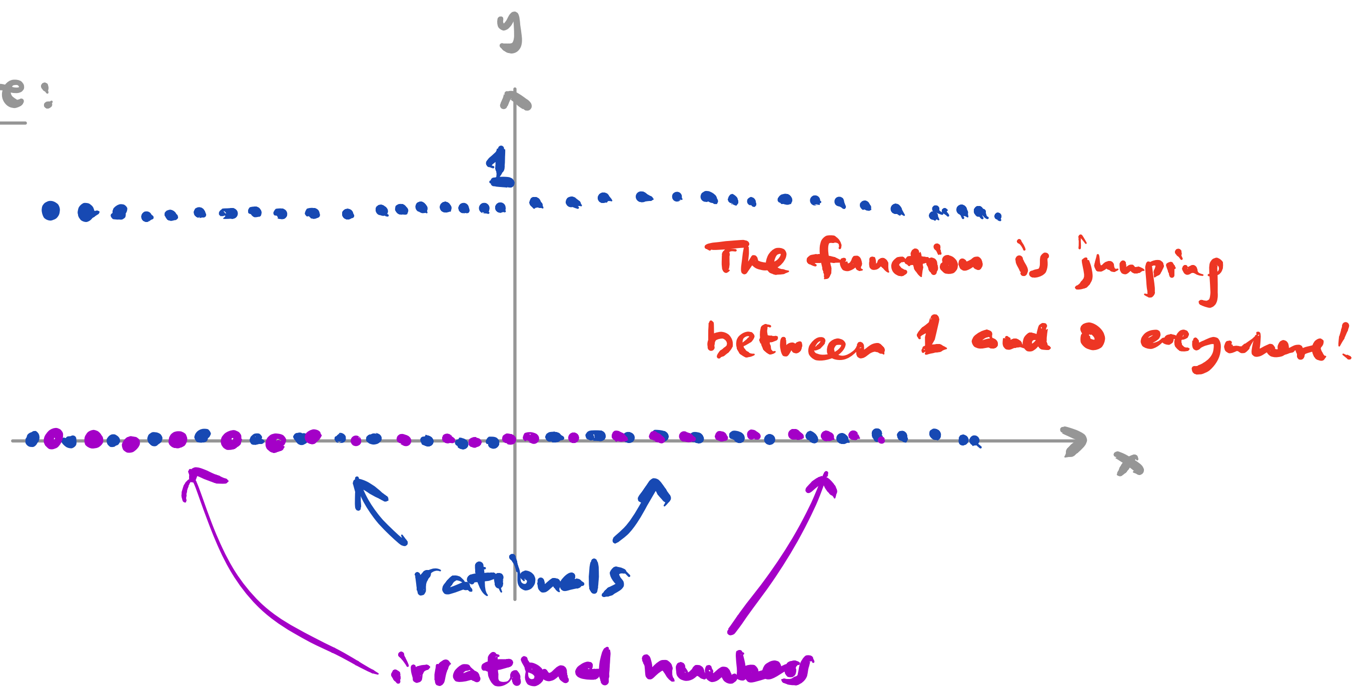
In the above example, the function is cts everywhere except at one point $c=0$, where the function "jump".

Example 2: The function $f: A = \mathbb{R} \to \mathbb{R}$

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is dis-continuous EVERYWHERE!

Picture:



Proof: Key: $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$

Pick an arbitrary $c \in \mathbb{R}$. By density of $\mathbb{Q}$ & $\mathbb{R} \setminus \mathbb{Q}$, we can always construct sequences:

(x_n) of rational number $\neq c$ st $\lim(x_n) = c$

& (y_n) of irrational number $\neq c$ st $\lim(y_n) = c$.

THEN.

$$\left.\begin{array}{l} (f(x_n)) = (1) \to 1 \\ \qquad\qquad\qquad\quad\ \ \neq \\ (f(y_n)) = (0) \to 0 \end{array}\right\}$$

by Seq. Criteria $\lim_{x \to c} f(x)$ does NOT exist.

$\square$